On Flapping Flight.

By MAURICE F. FITZGERALD, B.A., Professor of Engineering in Queen's College, Belfast.

(Communicated by Lord Rayleigh, O.M., F.R.S. Received July 27, 1909.)

This paper is supplementary to one by the same author published in the 'Proceedings of the Royal Society'* in 1899, "On Flapping Flight of Aeroplanes," communicated by the late Prof. George F. FitzGerald, F.R.S.

Towards the end of that paper, the author refers to the subject of hovering, and points out that the method of his paper does not apply, in any satisfactory way, to that case. The paper of 1899, in fact, considered the force supporting an aeroplane as due to its motion with nearly constant speed in a nearly horizontal, but slightly waved path, to which its plane was inclined at a small variable angle.

The present paper ignores speed of advance as a source of supporting pressure, and considers in its place the resistance opposed to the acceleration of a body immersed in a liquid, which may be represented by a virtual addition to its mass, without any corresponding addition to its weight. This virtual addition being, moreover, dependent on the direction of the acceleration, much greater, for example, for a flat disc subject to acceleration normal to its plane, than to edgewise, enables a wing to be treated as if it were capable of varying in mass without varying in weight, by suitable manœuvring.

It is assumed, in short, that the wing can shed, or escape from, the virtual added mass at will, as, for example, the paddle of a Canadian canoe can shed, or slip out of, the disturbance it has created during a stroke, when brought forward again by a sweeping motion without being lifted out of the water, leaving the disturbance to carry away momentum and energy in the wake.

It is evident that if the possibility of thus shedding impulsive disturbances be granted, a heavy body or engine attached to a wing (fig. 1) should be capable of supporting itself and the wing in the air in a succession of leaps, plotted on a time base in the figure, so as to keep its level on an average constant. For if the whole be supposed initially falling, a sufficiently energetic pull given by the engine, drawing the wing down and itself up, could (stage I) convert the fall of their common centre of gravity into a rise, if there were a weightless but massive body temporarily attached to the wing. If, when this rise had been started, the added mass were shed or escaped from, there would be an interval (stage II) in which the centre of gravity of

* Vol. 64, pp. 420—430.

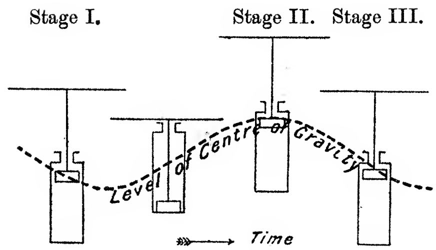
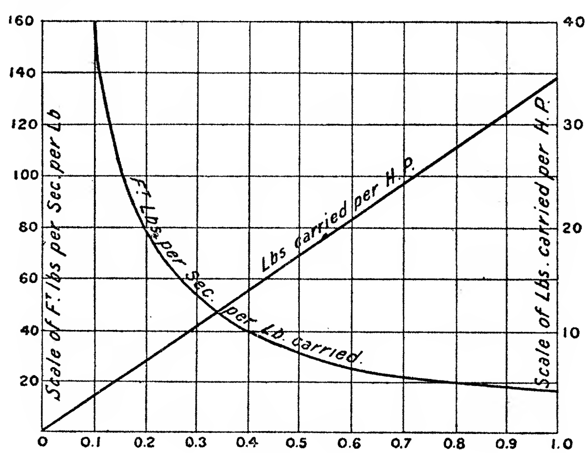


Fig. 1.

the whole would continue to rise, during which the wing and body could be replaced in their original relative positions by mutual action between them, wholly internal to their system, free except for gravity. If at the end of the rise added mass were again picked up, the system might fall to its initial level and velocity (stage III), the process becomes cyclic, and may be repeated *ad infinitum.*

The virtual addition to the mass of the wing is, for brevity, hereinafter called A, and is reckoned as units per unit of mass of the whole, wing and engine together. Fig. 2 shows (as hereinafter is proved) the foot-lbs. per second required per pound carried, and the pounds carried per horse-power, in cycles like that above described, at one flap per second, for certain values of A. No exact data are, in reality, available as to these values in particular cases, but the author thinks that those given cover a range likely to occur in air, with birds, insects, and such conceivable machines as the following.

Let the wing be a flat, circular disc, 12 feet diameter, and the body a three-horse engine.



Scale for A.

Fig. 2.—At one flap per second. At any other rate of flapping the weight carried per H.P. varies directly as the rate of flapping.

Lamb* gives the virtual increment of mass for a disc as $\frac{8}{3}\rho a^3$, in which a is the radius of the disc and ρ the density of the medium. This would make it, for the 12-foot disc in air, about 45 lbs., so that A = 45/weight of disc and body together. The total weight, allowing from 30 to 40 lbs. for the disc, and from 30 to 50 lbs. for the engine, etc., would be from 60 to 90 lbs. or more; so that A might have values ranging from 45/60 = 0·75, downwards. With A = 0·75, fig. 2 gives 26 lbs. carried at one flap per second per horse-power, so that three horse-power would more than suffice, but, at A = 0·5, only 17 lbs. are carried per horse-power, and the load is 90 lbs., consequently the rate of flapping would have to be raised from 60 to about 110 per minute for a three-horse engine to carry the load. In this matter, frequency plays exactly the same part, in economising power, as speed of advance would in the flight of aerodromes, worked under conditions satisfying Langley's law. The lines of fig. 2 and rule as to rate of flapping are arrived at as follows:—

If z be the height above a datum level of the wing, dz/dt its velocity, and d^2z/dt^2 its acceleration, all reckoned positive upwards, A the temporary virtual added mass attached to it, and f the pressure on it of the air, then f will, in common parlance, be described as the pressure due to A's inertia, and, whether A be varying or not, is $-\frac{d}{dt}\left(\frac{\mathrm{A}}{g}\frac{dz}{dt}\right)$, reckoning forces in pounds weight. When A is dropped or shed, its momentum is $\left(\frac{\mathrm{A}}{g}\frac{dz}{dt}\right)_{l\mathrm{T}}$, where the subscript lT denotes values of A and dz/dt at time lT, l being some proper fraction, and T the duration of one cycle, and by whatever way A comes to its then value, it was started from rest, piecemeal or otherwise, by the action of f, so that $\left(\frac{\mathrm{A}}{g}\frac{dz}{dt}\right)_{l\mathrm{T}}$ is the impulse of $-f$ during time lT, *i.e.* $= -\int_0^{l\mathrm{T}} f dt$. On the other hand, the impulse of f on the machine as a whole, during the same period, upwards, must balance that of gravity downwards during the whole cycle, which is 1 lb. weight for time T, so that $\int_0^{l\mathrm{T}} f dt = \mathrm{T}$. Consequently $\frac{dz}{dt_{l\mathrm{T}}} = -\frac{g\mathrm{T}}{\mathrm{A}}$. The kinetic energy carried away in A when it is shed is $\frac{\mathrm{A}}{2g}\left(\frac{dz}{dt_{l\mathrm{T}}}\right)^2$, which, putting $-\frac{g\mathrm{T}}{\mathrm{A}}$ for $\frac{dz}{dt}$, gives the external work done per cycle to carry 1 lb. $= g\mathrm{T}^2/2\mathrm{A}$, and the average rate of working over a series of cycles $= g\mathrm{T}/2\mathrm{A}$, on which fig. 2 and the frequency rule are based, since T varies inversely as the frequency.

In reality, this is the least rate of working possible, and more must be

* 'Hydrodynamics,' 3rd ed., p. 131.

done in any actual case, because any actual disturbance will contain a good deal of energy in the form of eddies, of which no account is here taken, but may perhaps be to some extent capable of estimation, in some cases, as if, for instance, A had to be conceived as dropped rotating, like the wake of a screw propeller. The author, being unable to form any general estimate of this, except by guess, has left out the item in fig. 2, to be supplied by better information. The value taken above for A in the case of the 12-foot disc is, of course, to be understood as merely an indication of the order of magnitude likely to be found in actual cases.

The author thinks that it may not be far from the truth to assume, for present purposes, that A may be treated as a constant in the kind of disturbances here contemplated, which are like those made by an oar, being started severally from rest by quick short impulses, and to a considerable extent isolated from one another. For if $f=\frac{d}{dt}(\mathrm{MV}) = \mathrm{M}\frac{d\mathrm{V}}{dt} + \mathrm{V}\frac{d\mathrm{M}}{dt}$ in general, and V be initially zero, and for a short time thereafter small, while M and $d\mathrm{V}/dt$ are not, the term $\mathrm{M}\frac{d\mathrm{V}}{dt}$ will predominate at first in disturbances started from rest, as above described, and A may be considered as practically constant during the creation of each, and as the same constant for a good many in succession also, because the general stream system surrounding them will alter the field in which successive ones are generated very slowly when the train of disturbances has become a long one. Taking A, then, as a constant during the period of its existence, the work done corresponding to $g\mathrm{T}^2/2\mathrm{A}$ by the pressure in the cylinder on the piston may be expressed as a function of ds/dt, the piston speed; of S, the part of the stroke made during $l\mathrm{T}$; of A; of T and l; and of the parts E pounds of engine, and $1-\mathrm{E}$ pounds of wing, into which the total weight of 1 lb. of the whole is divided. Using z, as before, for the height of the wing over datum, ζ for that of the body, h for that of their common centre of gravity, and P for the total pressure on the piston,

$$(1)\quad h = \mathrm{E}\zeta+(1-\mathrm{E})z; \qquad (2)\quad \frac{\mathrm{A}+1-\mathrm{E}}{g}\frac{d^2z}{dt^2} = -(1-\mathrm{E})-\mathrm{P};$$

$$(3)\quad \frac{\mathrm{E}}{g}\frac{d^2\zeta}{dt^2} = -\mathrm{E}+\mathrm{P}; \quad \text{and} \quad (4)\quad \frac{ds}{dt}=\frac{d\zeta}{dt}-\frac{dz}{dt}$$

is the relative velocity of body and wing. P is, in fig. 1, taken positive above the piston, and ds/dt, as taken in (4), gives $\mathrm{P}(ds/dt)$ positive when $d\zeta/dt$ and dz/dt are both positive, *i.e.* upwards, with $d\zeta/dt$ the greater,

so that the volume above the piston is increasing. On differentiating (1) it will be observed that

$$\frac{dh}{dt} = \mathrm{E}\frac{d\zeta}{dt} + (1-\mathrm{E})\frac{dz}{dt} \text{ is identical with } \frac{dh}{dt} = \frac{dz}{dt} + \mathrm{E}\frac{ds}{dt}.$$

Using (2) and (3) to find $\frac{d^2s}{dt^2} = \frac{d^2\zeta}{dt^2} - \frac{d^2z}{dt^2}$, P can be put in the form $\mathrm{P} = \frac{\mathrm{E}(1-\mathrm{E})}{g}\frac{d^2s}{dt^2} + \frac{\mathrm{AE}}{\mathrm{A}+1}\left(1+\frac{\mathrm{E}}{g}\frac{d^2s}{dt^2}\right)$. Multiplying this across by ds/dt and integrating all round the cycle, the first term on the right-hand side, which represents internal kinetic energy of the system, disappears, as ds/dt is to be cyclic, but the other one exists only during $l\mathrm{T}$, A being zero during the rest of the cycle; and the external work expressed by that integral on being equated to $g\mathrm{T}^2/2\mathrm{A}$ gives, S denoting, as above mentioned, the stroke during $l\mathrm{T}$.

$$(5)\quad \frac{\mathrm{AE}}{\mathrm{A}+1}\left\{\mathrm{S}+\mathrm{E}\frac{\mathrm{V}^2-\mathrm{U}^2}{2g}\right\} = \frac{g\mathrm{T}^2}{2\mathrm{A}},$$

where U and V stand for ds/dt_0 and $ds/dt_{l\mathrm{T}}$ respectively. U and V can be found separately; for $\frac{dh}{dt_{l\mathrm{T}}} - \frac{dh}{dt_0} = \frac{dz}{dt_{l\mathrm{T}}} + \mathrm{E}(\mathrm{V}-\mathrm{U})$ if $\frac{dz}{dt_0} = 0$, which it must be, to allow A to be picked up without having had imparted to it some extraneous impulse, and $\frac{dz}{dt_{l\mathrm{T}}} = -\frac{g\mathrm{T}}{\mathrm{A}}$, while $\frac{dh}{dt_{l\mathrm{T}}} - \frac{dh}{dt_0}$ is, by $\frac{dh}{dt}$ being cyclic, the same as $\frac{dh}{dt_{l\mathrm{T}}} - \frac{dh}{dt_\mathrm{T}}$ and this latter is $g\mathrm{T}(1-l)$, being the change of velocity of the common centre of gravity during the part of the cycle in which A does not exist. Dividing $\mathrm{V}^2-\mathrm{U}^2$, derived from (5), by $(\mathrm{V}-\mathrm{U})$ found as just described, $\mathrm{V}+\mathrm{U}$ is found, and V and U separated, giving

$$(6)\quad \begin{cases} \mathrm{U} = \dfrac{ds}{dt_0} = \dfrac{g\mathrm{T}}{2\mathrm{EA}}\left\{\dfrac{(\mathrm{A}+1)-\frac{2\mathrm{S}}{g\mathrm{T}^2}\mathrm{EA}^2-(\mathrm{A}+1-\mathrm{A}l)^2}{\mathrm{A}+1-\mathrm{A}l}\right\}, \\[2ex] \mathrm{V} = \dfrac{ds}{dt_{l\mathrm{T}}} = \dfrac{g\mathrm{T}}{2\mathrm{EA}}\left\{\dfrac{(\mathrm{A}+1)-\frac{2\mathrm{S}}{g\mathrm{T}^2}\mathrm{EA}^2+(\mathrm{A}+1-\mathrm{A}l)^2}{(\mathrm{A}+1-\mathrm{A}l)}\right\}. \end{cases}$$

Similarly, by twice integrating the equations of motion to find $h_{l\mathrm{T}}-h_0$, after eliminating P, and putting $h_{l\mathrm{T}}-h_0 = h_{l\mathrm{T}}-h_\mathrm{T}$, since h is cyclic, $= -\frac{1}{2g}\left\{\left(\frac{dh}{dt^{l\mathrm{T}}}\right)^2 - \left(\frac{dh}{dt_0}\right)^2\right\}$, which has a known factor $\left(\frac{dh}{dt_{l\mathrm{T}}} - \frac{dh}{dt_0}\right) = g\mathrm{T}(1-l)$, as above pointed out, $\frac{dh}{dt_{l\mathrm{T}}} + \frac{dh}{dt_0}$ is found, and the two velocities separated. As $dh/dt_0 = \mathrm{EU}$, a second expression appears for U, but, on reduction, is found identical with (6). As h is independent of P during A's non-

existence, so far as the equation of internal work is concerned, except with regard to terminal values, nothing more can be got, without more detailed specification of the cycle, and the part of P represented by $E(1-E)\frac{d^2s}{dt^2}$ might be supplied by pure mechanism started with a proper internal store of kinetic energy once for all.

With respect to specifying particular cycles, they must be cyclic but not necessarily periodic in form; $dz/dt_0 = 0$ as above pointed out, and in general $d^2z/dt^2 = 0$ at lT, and negative between $t = 0$ and $t = l\mathrm{T}$, because d^2z/dt^2 must be negative if $f = -\frac{\mathrm{A}}{g}\frac{d^2z}{dt^2}$ is to be a supporting force, and $\frac{d^2z}{dt^2}$ changes sign, in general, in passing through zero, so that f would become a depressing force unless A be then shed. A simple case is to assume $z = \mathrm{R}(\cos\Omega\, t - 1)$, which will satisfy the conditions if $l = \frac{1}{4}$, $\mathrm{R} = \mathrm{S} = g\mathrm{T}^2/2\pi\mathrm{A}$. It might be thought that some probable guesses at values of A might be made on some such argument as the following. Rooks and many other birds flap at a frequency of from 250 to 350 per minute, or thereabouts; $\frac{1}{2}g\mathrm{T}^2$ is the height of free fall in time T, and, allowing for the centre of pressure being somewhat within the wing tip, a rough guess at the value of R in birds of about their size would suggest some value about $\frac{1}{3}$ for A. But, in reality, such inferences are profoundly uncertain. Birds' flight is, no doubt, partly of the flapping character now investigated, but is also, no doubt, partly aerodromic, and this would have to be allowed for on the lines of the author's paper of 1899. Even if this were put aside, as in hovering in a calm, there is no sort of certainty that the cycle used is that assumed. There is a class of cycles wherein the apparent stroke is not limited by geometrical constraints in the ordinary sense; in which the wing and body may, for example, be relatively at rest in stage III, and there is no such direct relation between S and the apparent stroke as R = S above; in which in fact, the cycle could not be directly inferred from observation of the wing path, without possessing the equivalent of indicator diagrams of the muscular actions accompanying it.

Other points arise with respect to geometrically similar machines of the same materials, but differing in size. Suppose, for example, it had been ascertained that a herring gull used the simple harmonic cycle above mentioned, and he were taken as model for a machine (say) three times his size in linear dimensions, and twenty-seven times his weight. The A of the machine would be the same as that of the gull, since A varies as the cube of linear dimensions, like weight, not like wing area. To possess the same economy of power as the gull, the frequency of the machine would then

have to be the same as that of the gull and, as $R = g(T^2/2\pi A)$, the stroke of the machine would be, not the geometrical thrice, but the same as that of its prototype.

Insects and humming birds use such high frequencies, running into hundreds per second, that their horse-power per pound carried may be very low, not exceeding, perhaps, one-fourth of the rate per pound of their own weight which ordinary mammals exert; so small, in fact, that aeroplanes would have to be propelled at several thousand miles per hour, with direct head resistance of body and framework, etc., annulled, to equal it.

The author believes that, for physiological reasons, it is unlikely that birds can differ very greatly from one another in their rate of energy expenditure per pound of their own weight. It would, he thinks, be more probable that much of their apparently anomalous variety of wing action is due to differences in the value of A among them, leading to marked differences in frequency to compensate for those in A, while retaining a fairly constant efficiency. The frequency has little effect, as remarked in the paper of 1899* on the economy of power in the aerodromic flight there treated of.

In view of the circumstance that all cycles of the same period have the same efficiency for any given value of A, the author has not considered it necessary to enter into details as to the values of l, U, V, etc., occurring in particular cycles, or whether any limitations may arise, in particular cycles, among the possible values of A, E, and l. For any practical purposes, the successful shedding of A would be the only really important point, and the method of securing that would, he conceives, be determinable only by experiment.

The essential points as to which the author conceives this paper to be supplementary to that of 1899 are, that, in the impulsive flapping flight here treated, (I) the inertia effect due to acceleration of the wing affords a basis for a satisfactory explanation of how hovering and slow flight in a calm can be effected; (II) there is no special virtue in any particular kind of cycle, as such; (III) the average rate of expenditure of energy to carry a given weight varies, *cæteris paribus*, inversely as the frequency of flapping.

* Page 425, near foot.